Primitives par substitution ou changement de variables ; énoncés

$$\textbf{poser } t = f(x) \Rightarrow \frac{dt}{dx} = f'(x) \Leftrightarrow dt = f'(x).dx$$

1. $\int \frac{5x}{\sqrt{7+x^2}}dx$

2. $\int \frac{\cos x}{\sin^3 x}dx$

3. $\int \frac{7x}{\sqrt{5+x^2}}dx$

4. $\int \frac{\sin x}{\cos^3 x}dx$

5. $\int (x+1).\cos(x^2+2x-5).dx$

6. $\int \frac{x}{\sqrt{4-x^2}}dx$

7. $\int (x+4).\sin(x^2+8x-5).dx$

8. $\int \frac{x}{\sqrt{25-x^2}}dx$

9. $\int \frac{\arccos x}{\sqrt{1-x^2}}dx$

10. $\int \frac{\arcsin x}{\sqrt{1-x^2}}dx$

11. $\int \frac{5x}{(x^2-3)^2}.dx$

12. $\int (2x+3).\sin(2x^2+6x-5).dx$

13. $\int \frac{x}{\sqrt{9-3x^2}}dx$

14. $\int \frac{\operatorname{arctg} x}{1+x^2}dx$

15. $\int \sqrt{9-x^2}\,dx$

16. $\int \frac{2x-1}{x(x-1)}\,dx$

17. $\int \frac{1}{\left(\sqrt{1-x^2}\right)^3}\,dx$

18. $\int \frac{dx}{tgx+\sin x}$

19. $\int \frac{1}{x(x-1)}\,dx$

Primitives par substitution ou changement de variables : corrigés

$$\textbf{poser } t = f(x) \Rightarrow \frac{dt}{dx} = f'(x) \Leftrightarrow dt = f'(x).dx$$

1. $\mathbf{P = \int \frac{5x}{\sqrt{7+x^2}} dx}$

Posons $t = 7+x^2$ $\quad dt = 2x.dx \Leftrightarrow dx = \frac{dt}{2x}$

$$\mathbf{P} = \int \frac{5x}{t^{1/2}}.\frac{dt}{2x} = \frac{5}{2}\int t^{-1/2}dt = \frac{5}{2}\frac{t^{1/2}}{1/2} + C = 5\sqrt{t} + C = \mathbf{5\sqrt{7+x^2} + C}$$

2. $\mathbf{P = \int \frac{\cos x}{\sin^3 x} dx}$

Posons $t = \sin x$ $\quad dt = \cos x.dx \Leftrightarrow dx = \frac{dt}{\cos x}$

$$\mathbf{P} = \int \frac{\cos x}{t^3}\frac{dt}{\cos x} = \int t^{-3}.dt = \frac{t^{-2}}{-2} + C = -\frac{1}{2t^2} + C = \mathbf{-\frac{1}{2\sin^2 x} + C}$$

3. $\mathbf{P = \int \frac{7x}{\sqrt{5+x^2}} dx}$

Posons $t = 5+x^2$ $\quad dt = 2x.dx \Leftrightarrow dx = \frac{dt}{2x}$

$$\mathbf{P} = \int \frac{7x}{t^{1/2}}.\frac{dt}{2x} = \frac{7}{2}\int t^{-1/2}dt = \frac{7}{2}\frac{t^{1/2}}{1/2} + C = 7\sqrt{t} + C = \mathbf{7\sqrt{5+x^2} + C}$$

4. $\mathbf{P = \int \frac{\sin x}{\cos^3 x} dx}$

Posons $t = \cos x$ $\quad dt = -\sin x.dx \Leftrightarrow dx = \frac{dt}{-\sin x}$

$$\mathbf{P} = \int \frac{\sin x}{t^3}\frac{dt}{-\sin x} = -\int t^{-3}.dt = -\frac{t^{-2}}{-2} + C = \frac{1}{2t^2} + C = \mathbf{\frac{1}{2\cos^2 x} + C}$$

5. $\mathbf{P = \int (x+1).\cos(x^2+2x-5).dx}$

Posons $t = x^2 + 2x - 5$ $\quad dt = (2x+2)dx \Leftrightarrow dx = \frac{dt}{2(x+1)}$

$$\mathbf{P}= \int (x+1).\cos t.\frac{dt}{2(x+1)} = \frac{1}{2}\int \cos t.dt = \frac{1}{2}.\sin t + C = \mathbf{\frac{1}{2}\sin(x^2+2x-5)+C}$$

6. $\mathbf{P = \int \frac{x}{\sqrt{4-x^2}}dx}$

Posons $t = 4-x^2$ $\quad dt = -2xdx \Leftrightarrow dx = \frac{dt}{-2x}$

$$\mathbf{P}= \int \frac{x}{\sqrt{t}}.\frac{dt}{-2x} = -\frac{1}{2}\int t^{-1/2}.dt = -\frac{1}{2}.\frac{t^{1/2}}{1/2} + C = -\sqrt{t} + C = \mathbf{-\sqrt{4-x^2}+C}$$

7. $\mathbf{P = \int (x+4).\sin(x^2+8x-5).dx}$

Posons $t = x^2 + 8x - 5$ $\quad dt = (2x+8)dx \Leftrightarrow dx = \frac{dt}{2(x+4)}$

$$\mathbf{P}= \int (x+4).\sin t.\frac{dt}{2(x+4)} = \frac{1}{2}\int \sin t.dt = -\frac{1}{2}.\cos t + C = \mathbf{-\frac{1}{2}\cos(x^2+8x-5)+C}$$

8. $\mathbf{P = \int \frac{x}{\sqrt{25-x^2}}dx}$

Posons $t = 25-x^2$ $\quad dt = -2xdx \Leftrightarrow dx = \frac{dt}{-2x}$

$$\mathbf{P}= \int \frac{x}{\sqrt{t}}.\frac{dt}{-2x} = -\frac{1}{2}\int t^{-1/2}.dt = -\frac{1}{2}.\frac{t^{1/2}}{1/2} + C = -\sqrt{t} + C = \mathbf{-\sqrt{25-x^2}+C}$$

9. $\mathbf{P = \int \frac{\arccos x}{\sqrt{1-x^2}}dx}$

Posons $t = \arccos x$ $\quad dt = \frac{-1}{\sqrt{1-x^2}}dx \Leftrightarrow dx = -\sqrt{1-x^2}.dt$

$$\mathbf{P} = \int \frac{t}{\sqrt{1-x^2}}.\left(-\sqrt{1-x^2}\right)dt = -\int t.dt = -\frac{t^2}{2} + C = \mathbf{-\frac{\arccos^2 x}{2}+C}$$

10. $\mathbf{P = \int \frac{\arcsin x}{\sqrt{1-x^2}} dx}$

Posons t = arcsin x $\quad dt = \frac{1}{\sqrt{1-x^2}} dx \Leftrightarrow dx = \sqrt{1-x^2}.dt$

$$\mathbf{P} = \int \frac{t}{\sqrt{1-x^2}}.\left(\sqrt{1-x^2}\right)dt = \int t.dt = \frac{t^2}{2} + C = \mathbf{\frac{\arcsin^2 x}{2} + C}$$

11. $\mathbf{P = \int \frac{5x}{(x^2-3)^2} .dx}$

Posons t = x²-3 $\quad dt = 2x.dx \Leftrightarrow dx = \frac{dt}{2x}$

$$\mathbf{P} = \int \frac{5x}{t^2}.\frac{dt}{2x} = \frac{5}{2}\int t^{-2}.dt = \frac{5}{2}.\frac{t^{-1}}{-1} + C = -\frac{5}{2.t} + C$$

$$\mathbf{= -\frac{5}{2(x^2-3)} + C}$$

12. $\mathbf{P = \int (2x+3).\sin(2x^2+6x-5).dx}$

Posons t = 2x²+6x-5 $\quad dt = (4x+6)\, dx \Leftrightarrow dx = \frac{dt}{2(2x+3)}$

$$\mathbf{P} = \int (2x+3).\sin t.\frac{dt}{2(2x+3)} = \frac{1}{2}\int \sin t.dt = \frac{-1}{2}\cos t + C$$

$$\mathbf{= \frac{-1}{2}\cos(2x^2+6x-5) + C}$$

13. $\mathbf{P = \int \frac{x}{\sqrt{9-3x^2}} dx}$

Posons t = 9-3x² $\quad dt = (-6x)\, dx \Leftrightarrow dx = \frac{dt}{-6x}$

$$\mathbf{P} = \int \frac{x}{\sqrt{t}}.\frac{dt}{-6x} = \frac{-1}{6}\int t^{\frac{-1}{2}}.dt = \frac{-1}{6}\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{-1}{3}\sqrt{t} + C$$

$$\mathbf{= -\frac{1}{3}\sqrt{9-3x^2+C}}$$

14. $\displaystyle\int \frac{\mathbf{arctgx}}{\mathbf{1+x^2}}\mathbf{dx}$

Posons t = arctgx $\qquad dt = \dfrac{1}{1+x^2}dx \Leftrightarrow dx = (1+x^2)dt$

$$\mathbf{P}= \int \frac{t}{1+x^2}(1+x^2)dt = \frac{t^2}{2}+C$$

$$= \frac{\mathbf{(arctg\,x)^2}}{\mathbf{2}} + \mathbf{C}$$

15. $\mathbf{P = \int \sqrt{9-x^2}\,dx}$

Posons $x = 3\sin t \left(\Leftrightarrow t = \arcsin\dfrac{x}{3} : t \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]\right) \qquad dx = 3\ \cos t\ dt$

$$P = \int \sqrt{9-9\sin^2 t}.3.\cos t.dt = 3\int \sqrt{9(1-\sin^2 t)}.\cos t.dt = 9\int \cos^2 t.dt$$

Formule :

$$\mathbf{1+\cos x = 2\cos^2\frac{x}{2} \Leftrightarrow \cos^2\frac{x}{2} = \frac{1}{2} + \frac{\cos x}{2}}$$

$$P = \frac{9}{2}\int(1+\cos 2t)dt = \frac{9}{2}\left(t + \frac{\sin 2t}{2}\right) + C$$

Calculs :

- $\sin 2t = 2\sin t\cos t = 2.\dfrac{x}{3}\cos t$
- $\sin^2 t + \cos^2 t = 1 \Leftrightarrow \cos^2 t = 1-\sin^2 t \Leftrightarrow \cos^2 t = 1-\dfrac{x^2}{9}$

  $\Leftrightarrow \cos t = \sqrt{1-\dfrac{x^2}{9}}\ :\ \cos t > 0 \text{ car } t \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$
- $\sin 2t = 2.\dfrac{x}{3}.\sqrt{1-\dfrac{x^2}{9}}$

$$\mathbf{P = \frac{9}{2}\left(\arcsin\frac{x}{3} + \frac{x}{3}\sqrt{1-\frac{x^2}{9}}\right) + C = \frac{9}{2}\left(\arcsin\frac{x}{3} + \frac{x\sqrt{9-x^2}}{2}\right) + C}$$

16. $\mathbf{P = \int \frac{2x-1}{x(x-1)}dx}$

$$P = \int \frac{2x-1}{x^2 - x}dx$$

Posons $t = x^2 - x$ $\quad dt = (2x-1)dx \Leftrightarrow dx = \frac{dt}{2x\text{ -}1}$

$$\mathbf{P = \int \frac{2x-1}{t} \cdot \frac{dt}{2x-1} = \int \frac{dt}{t} = \ln|t| + C}$$

$$\mathbf{P = \ln|x^2 - x| + C}$$

17. $\mathbf{P = \int \frac{1}{\left(\sqrt{1-x^2}\right)^3}dx}$

Posons $x = \sin t \left(\Leftrightarrow t = \arcsin x : t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$ $\quad dx = \cos t\, dt$

$$P = \int \frac{1}{\left(\sqrt{1-\sin^2 t}\right)^3}\cos t.dt = \int \frac{1}{\cos^3 t}\cos t.dt = \int \frac{1}{\cos^2 t}dt = tgt + C$$

<u>Calculs</u> :

- $tgt = \frac{\sin t}{\cos t} = \frac{x}{\cos t}$
- $\sin^2 t + \cos^2 t = 1 \Leftrightarrow \cos t = \sqrt{1-\sin^2 t} = \sqrt{1-x^2}$ $\quad \cos t > 0$ car $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $tgt = \frac{x}{\sqrt{1-x^2}}$

$$\mathbf{P = \frac{x}{\sqrt{1-x^2}} + C}$$

18. $\mathbf{P = \int \frac{dx}{tgx + \sin x}}$

Posons : $t = tg\frac{x}{2}$ $\quad dt = \left(1 + tg^2\frac{x}{2}\right).\frac{1}{2}.dx \Leftrightarrow dt = \frac{1+t^2}{2}dx \Leftrightarrow dx = \frac{2.dt}{1+t^2}$

<u>Formules</u> :

$$\mathbf{\sin x = \frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}} = \frac{2t}{1+t^2}} \qquad \mathbf{tgx = \frac{2tg\frac{x}{2}}{1-tg^2\frac{x}{2}} = \frac{2t}{1-t^2}}$$

$$P = \int \frac{2.dt}{(1+t^2)\left(\frac{2t}{1-t^2} + \frac{2t}{1+t^2}\right)} = 2\int \frac{dt}{(1+t^2)\left(\frac{2t+2t^3+2t-2t^3}{(1-t^2)(1+t^2)}\right)}$$

$$= 2\int \frac{1-t^2}{4t}dt = \frac{1}{2}\int \left(t^{-1} - t\right)dt = \frac{1}{2}\left(\ln|t| - \frac{t^2}{2}\right) + C$$

$$= \frac{1}{2}\left(\ln\left|tg\frac{x}{2}\right| - \frac{tg^2\frac{x}{2}}{2}\right) + C$$

$$\mathbf{P = \frac{1}{2}\left(\ln\left|tg\frac{x}{2}\right| - \frac{tg^2\frac{x}{2}}{2}\right) + C}$$

19. $\mathbf{P = \int \frac{1}{x(x-1)}dx}$

Posons : $t = \frac{x-1}{x}\ (\Rightarrow x - 1 = x.t)$ $\quad dt = \frac{1}{x^2}dx \Leftrightarrow dx = x^2dt$

$$\mathbf{P = \int \frac{1}{x.x.t}x^2dt = \int \frac{dt}{t} = \ln|t| + C}$$

$$\mathbf{P = \ln\left|\frac{x-1}{x}\right| + C}$$